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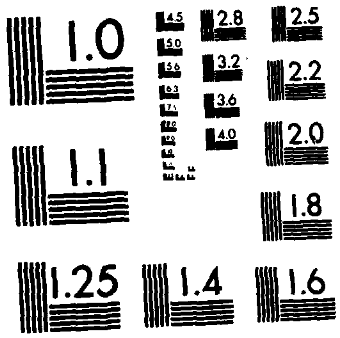


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POLE PLACEMENT FOR TIME

PERIODIC SYSTEMS

THESIS

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# POLE PLACEMENT FOR TIME PERIODIC SYSTEMS

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science



by
Carl S. Wolf
Capt USAF
Graduate Aeronautical Engineering

December 1982

## Acknowledgements

I would like to express my appreciation to Dr. William E. Wiesel for the direction and assistance he provided in this study. Dr. Wiesel has my continuing gratitude and respect for the help and depth of knowledge he contributed.

Captain Frank M. DeArmond deserves my thanks for the help he gave in day-to-day discussion of my work and his parallel effort. His assistance led to a closer understanding of the problem and the implications of its solution.

I owe a special debt to my wife, Lois, for her love and support throughout my course of study at the Air Force Institute of Technology.

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## List of Symbols

| | |
|---|---|
| $A(t)$ | coefficient matrix of the variational system |
| $B$ | control matrix |
| $C$ | monodromy matrix |
| $D$ | diagonal matrix of singular values |
| $\delta\bar{x}$ | variational state vector |
| $e$ | exponentation |
| $\bar{\eta}$ | modal vector |
| $F(t)$ | matrix |
| $\bar{f}(t)_i$ | the $i^{th}$ column of matrix F |
| $G(t)$ | gain matrix |
| $\bar{g}(t)_i$ | the $i^{th}$ column of matrix G |
| $I$ | identity matrix |
| $J$ | matrix of eigenvalues in Jordan conical form |
| $K$ | pole placement matrix |
| $L$ | lower triangular matrix |
| $L_1$-$L_5$ | Earth-Moon system Lagrangian points |
| $L'$ | augmented L matrix |
| $LE$ | matrix of left eigenvectors |
| $\bar{LE}_j$ | the $j^{th}$ column of LE |

## List of Symbols (Cont'd.)

| | |
|---|---|
| $\ell n$ | natural logarithm |
| $\Lambda(t)$ | matrix of solution vectors |
| $\lambda_i$ | the $i^{th}$ characteristic multiplier |
| $\omega_i$ | imaginary part of the $i^{th}$ complex eigenvalue |
| $\Phi(t)$ | fundamental matrix |
| $\Phi(t,t_0)$ | state transition matrix |
| m | number of columns |
| n | number of rows |
| Q | matrix of eigenvectors |
| $\bar{Q}_i$ | the $i^{th}$ eigenvector |
| R | matrix of Poincaré exponents |
| RE | matrix of right eigenvectors |
| $\bar{RE}_j$ | the $j^{th}$ column of RE |
| r | rank of matrix |
| $\rho_i$ | the $i^{th}$ Poincaré exponent |
| $\sigma_i$ | the $i^{th}$ characteristic multiplier |
| t | time |
| $t_0$ | initial time |
| T | period of oscillation |

## List of Symbols (Cont'd.)

| | |
|---|---|
| $\theta$ | augmented singular value matrix |
| $\bar{u}$ | upper triangular matrix |
| $U$ | upper triangular matrix |
| $U'$ | augmented U matrix |
| $\zeta_i$ | the $i^{th}$ singular value |

## Abstract

The technique of pole placement was applied to a time varying periodic system. When the control matrix was of full rank, unstable real roots were predictably moved to stable locations. Specified damping was successfully applied to oscillatory roots. In the case of the control matrix having less than full rank a least squares gain was applied. This minimized the error between all desired pole locations and controlled pole locations and as a consequence, prediction of final pole location was not possible. An exact method for predictably placing poles as desired was restricted by a controllability condition.

The method was applied to the restricted four-body problem and control of a satellite in a periodic orbit about the Earth-Moon Lagrangian point, $L_3$. A stabilized orbit was obtained for a model which allowed instantaneous control of both position and velocity. Stabilization was not successful in the case of control of velocity only. The least squares solution effected several modes simultaneously. The exact solution for gain was unobtainable as the controllability condition was not met.

# I. INTRODUCTION

## Background

The phenomenon of gravitational interactions between satellites and celestial bodies has long been a point of interest. In the past several decades the understanding of this phenomenon has become a requirement for satellite control. In nature, interactions between multiple large body systems can generate points of collected small objects. An example is the Trojan objects in the Sun-Jupiter system. The phenomenon is the physical realization of the periodic n-body problem in dynamics.

The Trojan objects were predicted by the solutions of the restricted three-body problem as put forth by Lagrange (1772). In this problem, three bodies exhibit coplanar motion. One has mass so small its motion does not effect the motion of the remaining two; this motion being circular orbits rotating at a constant rate about the barycenter. Lagrange found five equilibrium solutions to the equations of motion. These were points where gravitational forces on the small third body, as exerted by the other two, would balance the "centrifugal" force, thus leaving it fixed with respect to a reference frame rotating with the two large bodies. Figure 1 presents the rotating frame for the Earth-Moon system. A body put at one of the Libration points with zero relative velocity would stay there until perturbed. Gyldén called the Lagrange solutions "centers of Libration" since in this rotating frame the third body may also describe small time-periodic orbits

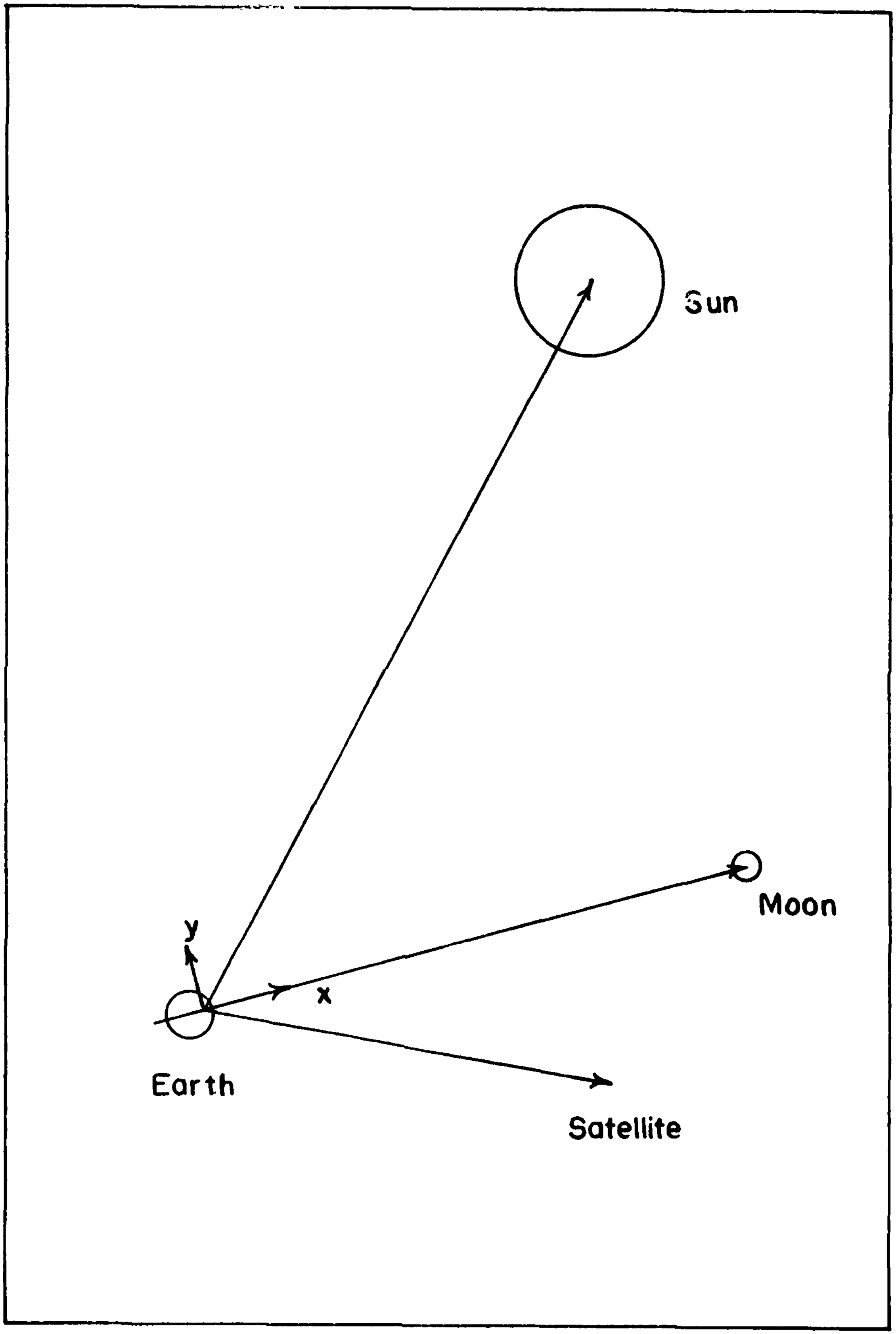


Figure 1 Geometry of the Four-Body Problem

about the fixed points. The equilibrium positions have been designated $L_1$ thru $L_5$ and are shown in figure 2 for the Earth-Moon system. The rotating frame has its origin at the Earth-Moon center of mass. The points $L_1$ to $L_3$ are collinear with the Earth-Moon line of sight while $L_4$ and $L_5$ each make an equilateral triangle with the Earth and Moon forming two vertices. Motion near the collinear points has been shown to be unstable through infinitesimal analysis (Ref 5:420-428). Unstable motion infers that a body at these points would depart if perturbed. The triangular points show infinitesimally stable motion in their neighborhood from a similar analysis, meaning the bodies at $L_4$ and $L_5$ would remain in that vicinity even when perturbed. Discovery of the Trojan objects which appear to librate near the Sun-Jupiter Lagrange points gives impetus to this result.

In the literature, distinction is made between stability at the libration point itself and stability of the periodic solutions about the point, if they exist (Ref 12:215-217). The fundamental questions at hand about motion near the Lagrange points concern the existence of periodic orbits and their stability. For the restricted three-body problem, these questions have been answered (Ref 5,8,12,14). Unfortunately, in implementing an actual satellite the use of the restricted three-body problem solutions in a control scheme does not suffice. Addition of a fourth body alone without effects of eccentricity or orbital inclination render the triangular points unstable.

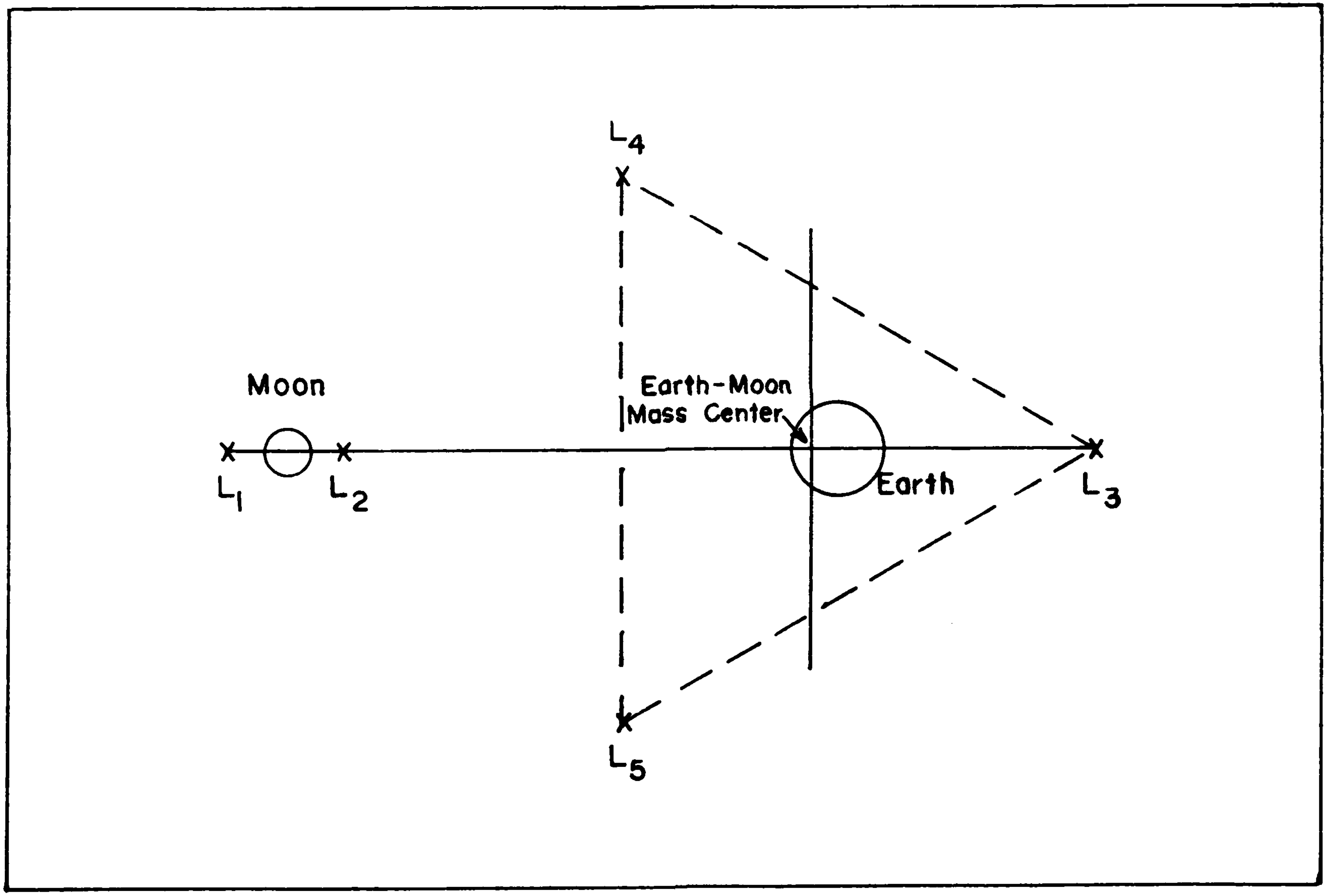


Figure 2  Lagrange Points in Earth-Moon System

The unstable $L_3$ orbit is made more unstable in this restricted four-body problem.

Militarily, the stability and control of periodic systems, specifically the libration point orbits has moved from the realm of academic interest to a requirement for the continued secure use of space for surveillance, early warning, command, communication, and control. Current satellites for those purposes are located in low and near Earth orbits. The Soviet Union has repeatedly demonstrated its ability to intercept and destroy low Earth orbit satellites. Recent revelations have indicated the Soviets now possess the capability to attack even satellites in geosynchronous orbit using an ICBM as a direct ascent attack vehicle (Ref 11). The move to the libration points orbits would have not only the advantage of increasing intercept time-of-flight from hours to days but has the advantage of increasing the area of coverage of each satellite. Synchronous satellites in orbit about the Earth's equator are at an altitude such that the North Pole is over the horizon and hidden from view. Thus, this is a blind spot in a surveillance net of geosynchronous satellites requiring many low orbit satellites with high orbital inclinations for continuous coverage instead. However, the libration point orbits are in the Earth-Moon orbital plane and hence at any time one satellite can have a direct line of sight to the Pole. Also, with satellites located at the libration points, global communications would be possible through one or more relays to these satellites. Thus, we have a vested interest to understand and solve the problem of control of periodic

systems.

Problem and Scope

The thrust of this effort is the control of unstable modes of a periodic system. A controllability statement on oscillatory time varying modes and unstable real modes will be attempted. By this we want to state under what conditions can we "stabilize" the system and arrange system poles predictably. To do this, an attempt will be made to adapt pole placement techniques for linear constant-coefficient systems to linearized periodic systems. The use of singular value decomposition along with techniques of generalized inverses will be explored to find the appropriate system gain required to stabilize the system. The restricted four-body problem will be used as a practical application for the techniques developed.

Previous work has centered on efficiently controlling a satellite affected by the unstable real mode of the $L_3$ periodic orbit as developed by Wiesel (Ref 16). The length of time control has been effective at reasonable cost has been increased for the restricted four-body problem but still is not long enough for practical application. Smith (Ref 10) applied linear, constant gain, feedback control to a satellite near the orbit calculated by Wiesel. The control law he developed was inadequate based upon the failure of various gains and feedback quantities used to return the satellite to the reference orbit economically. Shelton's (Ref 9) use of modal control allowed satellite control for up to two years

but when the perturbation effects of the actual Earth-Moon-Sun system were included his control law was inadequate. The Hamiltonian developed by Shelton and its subsequent equations of motion and equations of variation were used here.

## II. FLOQUET THEORY

The Hamiltonian for the four-body problem devised in reference 9 results in equations of motion which are explicit functions of time. The variational equations of this nonautonomous system can be written in linearized state-variable form

$$\delta\dot{\bar{x}} = [A(t)]\delta\bar{x} \tag{1}$$

The elements of the nxn matrix $A(t)$ are continuous periodic functions of time with period T such that the periodicity is expressed as

$$[A(t)] = [A(t+T)] \tag{2}$$

As a consequence of this periodicity, we will be able to predict motion for all time once the solution for system motion is found over one period. To do this requires using Floquet's Theorem. This theorem concerns the representation of a fundamental system of solutions of (1) as the product of a periodic matrix (with period T) and a solution matrix for a system with constant coefficients. The fundamental solution matrix has been given the symbol $\Phi(t)$ and can be expressed as

$$[\Phi(t)] = [Q(t)]e^{t[R]} \tag{3}$$

Here, $Q(t)$ is periodic where

$$[Q(t)] = [Q(t+T)] \tag{4}$$

and the columns of $\Phi(t)$ are related to the columns of $Q(t)$ by $e^{t[R]}$; R a constant matrix. The fundamental matrix satisfies

$$[\dot{\Phi}(t)] = [A(t)]\,[\Phi(t)] \tag{5}$$

If $\Phi(t+T)$ satisfies

$$[\dot{\Phi}(t+T)] = [A(t+T)]\,[\Phi(t+T)]\ , \tag{6}$$

then by (2), it also satisfies

$$[\dot{\Phi}(t+T)] = [A(t)]\,[\Phi(t+T)] \tag{7}$$

and is therefore itself a fundamental matrix. Since both $\Phi(t)$ and $\Phi(t+T)$ are solution matrices containing n-linearly independent solutions to (1), the columns of the matrices must be related such that

$$[\Phi(t+T)] = [\Phi(t)]\,[C] \tag{8}$$

where the matrix C is called the monodromy matrix. For t=0, equation (8) can be rearranged to

$$[\Phi^{-1}(0)]\ [\Phi(T)] = [C] \tag{9}$$

which will be used later. The monodromy matrix is not unique but depends upon the particular fundamental matrix used to satisfy (5). A matrix $\Phi(t)_p$ which satisfies (5) but also is the identity matrix at t=0 is called the principal fundamental matrix. The principal fundamental matrix gives

$$[\Phi(T)_p] = [C] \tag{10}$$

using (9) and from the fact that the inverse of an identity matrix is itself.

We now evaluate equation (3) at t=0 and take the inverse of the resulting matrix equation. Combining the outcome of this with equation (3), evaluated now at t=T, we get

$$[\Phi^{-1}(0)]\ [\Phi(T)] = [Q^{-1}(0)]\ [Q(T)]e^{T[R]} \tag{11}$$

Since Q(t) satisfies (4) the product $[Q^{-1}(0)]\ [Q(T)]$ reduces to the identity matrix and by invoking (9), equation (11) is reduced to

$$[C] = e^{T[R]} \tag{12}$$

and

$$[R] = \frac{1}{T}\ \ell n\ [C] \tag{13}$$

The eigenvalues of the C matrix are called the characteristic multipliers of A(t) and are symbolized by $\lambda_i (i=1,2,\ldots,n)$. The characteristic multipliers are related to the R matrix eigenvalues, $\rho$, by

$$\rho_i = \frac{1}{T}\ n\ (\lambda_i) \tag{14}$$

The $\rho_i$ are alternately called the characteristic or Poincaré exponents. The Poincaré exponents indicate the system stability with positive real (or real part for complex) eigenvalues being unstable. Since any monodromy matrix for a particular problem is similar to all others (Ref 5:266) and hence has the same eigenvalues, the characteristic multipliers as well as the characteristic exponents can be put in Jordan canonical form. When all the eigenvalues of a matrix are unique, the Jordan form is

$$[J] = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & \lambda_n \end{bmatrix} \tag{15}$$

In the case where some of the eigenvalues are complex conjugate pairs, a modified block diagonal form of the Jordan matrix is used. This allows the creation of an eigenvector corresponding to the real part of the oscillatory mode and an eigenvector for the imaginary part, replacing the two complex eigenvectors of the conjugate roots. The block diagonal [J] matrix is of the form

$$[J] = \begin{bmatrix} \sigma_1 & -\omega_1 & \cdot & \cdot & \cdot & 0 \\ \omega_1 & \sigma_1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_3 & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & \lambda_n \end{bmatrix} \tag{16}$$

where $\sigma$ is the real part and $\omega$ the imaginary part of the complex root.

There are both real and complex Poincaré exponents for the periodic orbit about $L_3$ (see Table I) so (16) depicts the Jordan form of R. Using this in (3) and (5) we get

$$[\dot{Q}(t)]e^{t[J]} + [Q(t)]\,[J]e^{t[J]} = [A(t)]\,[Q(t)]e^{t[J]} \tag{17}$$

TABLE I

Normalized Poincaré Exponents of Shelton's Orbit

| Mode | Poincaré | | Exponent |
|---|---|---|---|
| Planar Mode | 0.0 | + | 0.077738i |
| | 0.0 | - | 0.077738i |
| Planar Mode | 0.17893 | + | 0.0i |
| | -0.17893 | + | 0.0i |
| Out of Plane | 0.0 | + | 0.084134i |
| | 0.0 | - | 0.084134i |

The $e^{t[J]}$ terms are removed leaving

$$[\dot{Q}(t)] + [Q(t)]\,[J] = [A(t)]\,[Q(t)] \tag{18}$$

which can be solved for [J]

$$[J] = [Q^{-1}(t)]\left\{[A(t)]\,[Q(t)] - [\dot{Q}(t)\,]\right\} \tag{19}$$

Now, defining $\Lambda(t)$ as Q(t) partitioned into columns

$$[\Lambda(t)] = [\bar{Q}_1(t)\ ,\ \bar{Q}_2(t),\ldots,\bar{Q}_n(t)] \tag{20}$$

and putting this into (19) gives

$$[J] = [\Lambda^{-1}(t)] \left\{[A(t)]\,[\Lambda(t)] - [\dot{\Lambda}(t)]\right\} \tag{21}$$

The matrices $\Lambda$ and $\Lambda^{-1}$ are periodic with period, T. Remembering the linearization of the variational equations we had

$$\delta\bar{x}(t) = \bar{x}(t) - \bar{x}_0(t) = \left.\frac{\partial\bar{x}(t)}{\partial\bar{x}(t_0)}\right|_{\bar{x}_0(t)} \delta\bar{x}(t_0), \tag{22}$$

stating the variations were the difference between the actual and a reference solution. The reference solution used here being the Weisel periodic orbit. Defining

$$[\phi(t,t_0)] = \left.\frac{\partial\bar{x}(t)}{\partial\bar{x}(t_0)}\right|_{\bar{x}_0(t)} \tag{23}$$

we get (22) into the form

$$\delta\bar{x}(t) = [\phi(t,t_0)]\delta\bar{x}(t_0) \tag{24}$$

where $\phi(t,t_0)$ is the state transition matrix having the properties

$$\begin{aligned} [\phi(t,t_0)] &= [\phi(t,t_1)]\,[\phi(t_1,t_0)] \\ [\phi(t_0,t_0)] &= [I]. \end{aligned} \tag{25}$$

Since $\Phi(t)$ are solutions to (1) as is $\delta\bar{x}(t)$, (24) becomes

$$[\Phi(t)] = [\Phi(t,t_0)]\,[\Phi(t_0)]\ , \quad (26)$$

which, for $t_0=0$ and $t=T$ we get

$$[\Phi(T)] = [\Phi(T,0)]\,[\Phi(0)] \quad (27)$$

solving for $\Phi(T,0)$ and combining with equation (9) we can show

$$[C] = [\Phi(T,0)] \quad (28)$$

An arbitrary solution to (1), which is a column from (3), can be written as

$$\delta\bar{x}_i(t) = \bar{Q}_i(t)e^{\rho_i t} \quad (29)$$

Equation (24) at $t_0=0$ and (29) equate such that

$$\delta\bar{x}(t) = \bar{Q}_i(t)e^{\rho_i t} = [\Phi(t,0)]\delta\bar{x}_i(0). \quad (30)$$

Now, with t=0 in (29)

$$\delta\bar{x}_i(t) = \bar{Q}_i(t)e^{\rho_i t} = [\Phi(t,0)]\bar{Q}_i(0) \quad (31)$$

and again using (4)

$$\delta\bar{x}_i(T) = \bar{Q}_i(T)e^{\rho_i T} = \bar{Q}_i(0)e^{\rho_i T} = \phi(T,0)\bar{Q}_i(0) \quad (32)$$

at the end of one period.

Equation (32) is reduced to an eigenvalue eigenvector problem using (28)

$$\left\{[C] - e^{\rho_i t}\,[I]\right\}\bar{Q}_i(0) = \bar{0} \quad (33)$$

where the terms $e^{\rho_i t}$ are the eigenvalues and $\bar{Q}_i(0)$ the eigenvector. To get a complete solution the eigenvector must be integrated over an entire period. This is done by getting $\dot{\Lambda}$ on the left-hand side of (21). We can then solve (21) numerically since it is really an ordinary differential equation in $\Lambda(t)$ where these are just the partitioned Q(t) and its initial conditions are the eigenvectors of C. The values of $\Lambda^{-1}(t)$ are obtained directly from $\Lambda(t)$ and $\Lambda(t)$ is composed of n-linearly independent vectors, is of full rank and is invertible. Again, since the $\Lambda$ and $\Lambda^{-1}$ matrices are periodic, once the solution is obtained over one period it is known for all time.

# III. POLE PLACEMENT

Pole placement in linear constant coefficient systems allows the designer to specify system time response through the relocation of specific system modes or poles. Alteration of the time response is limited by system zeroes. This is accomplished by selective application of gain in the control law. The technique leaves non-specified modes unchanged. The technique can be applied to any controllable mode, hence the requirement for a controllability statement. Also, the system can be stabilized only if uncontrollable modes are stable. Stabilization is accomplished by moving poles from the right-half of the complex plane (unstable) to the left-half plane. Damping may be added to a complex pair of roots (again moving the roots to the left) but this must be accomplished by moving pairs of complex conjugates (Ref 3:362).

For Hamiltonian systems, the Poincaré exponents are always positive/negative pairs. This tells us that if we have a real root in the left-half plane its conjugate is in the right-half plane and the system is unstable. Complex roots also appear as conjugates. The problem at hand has two oscillatory modes and a real mode. The real mode lies in the orbital plane and has one unstable root. The oscillatory modes, one mode in and one mode out of the plane, are purely imaginary (marginally stable) in the restricted four-body problem but become unstable in more accurate models.

The technique of pole placement will be used to move the unstable real root to a specific point in the left-half plane along with adding damping to the oscillatory modes.

Implementation of pole placement for control requires augmenting the state equation such that

$$\delta\dot{\bar{x}}(t) = [A(t)]\delta\bar{x}(t) + [B(t)]\bar{u}(t) \tag{34}$$

Here, the matrix B applies control to the desired states and u is the control vector. The system requires transformation to a new, uncoupled, set of coordinates to insure any control applied has only the desired effect, specifically, leaving the stable real pole unaltered. This new set of modal variables is generated by

$$\bar{\eta} = [\Lambda^{-1}(t)]\delta\bar{x}(t). \tag{35}$$

The state vector is transformed by solving (35) for $\delta\bar{x}(t)$ and taking its time derivative, yielding

$$\delta\dot{\bar{x}}(t) = [\dot{\Lambda}(t)]\bar{\eta}(t) + [\Lambda(t)]\dot{\bar{\eta}}(t) \tag{36}$$

letting (34) equal (36) and again using the solution of (35) for $\delta\bar{x}(t)$

we have

$$[\dot{\Lambda}(t)]\bar{\eta}(t) + [\Lambda(t)]\dot{\bar{\eta}}(t) = [A(t)]\,[\Lambda(t)]\bar{\eta}(t) + [B(t)]\bar{u}(t) \quad (37)$$

which is readily solved for $\dot{\bar{\eta}}(t)$

$$\dot{\bar{\eta}}(t) = [\Lambda^{-1}(t)]\left\{[A(t)]\,[\Lambda(t)] - [\dot{\Lambda}(t)]\right\}\bar{\eta}(t) + [\Lambda^{-1}(t)]\,[B(t)]\bar{u}(t) \quad (38)$$

Remembering equation (18) and equation (20)

$$[\dot{\Lambda}(t)] = [A(t)]\,[\Lambda(t)] - [\Lambda(t)]\,[J] \quad (39)$$

and by defining

$$\bar{u}(t) = [G(t)]\,[\Lambda(t)]\,\bar{\eta}(t) \quad (40)$$

we derive

$$\dot{\bar{\eta}}(t) = [J]\,\bar{\eta}(t) + [\Lambda^{-1}(t)]\,[B(t)]\,[\Lambda(t)]\bar{\eta}(t) \quad (41)$$

Upon rearranging, this becomes

$$\dot{\bar{\eta}}(t) = \left\{[J] + [\Lambda^{-1}(t)]\,[B(t)]\,[G(t)]\,[\Lambda(t)]\right\}\bar{\eta}(t) \quad (42)$$

$$[K] = [J] + [\Lambda^{-1}(t)]\,[B(t)]\,[G(t)]\,[\Lambda(t)] \tag{43}$$

we write

$$\dot{\bar{\eta}}(t) = [K]\bar{\eta}(t) \tag{44}$$

The crux of the problem is we have picked K, the location of system poles and must now solve for the gain matrix, G, that gives us this desired response. This is done by first obtaining

$$[B]\,[G] = [\Lambda(t)]\left\{[K] - [J]\right\}[\Lambda^{-1}(t)] \tag{45}$$

then solving for the gain.

The matrices K, J, and $\Lambda$ are all known. The difficulty arises in finding the solution to (45) when B is singular or is a nonsquare matrix. The first problem we will address is the one where we have a square B matrix of full rank. Since this matrix is of full rank it is non-singular and invertible. The solution for the gain matrix can be found directly from

$$[G(t)] = [B^{-1}(t)]\,[\Lambda(t)]\left\{[K] - [J]\right\}[\Lambda^{-1}(t)] \tag{46}$$

The system in this case is always controllable since the product [B] [G] in equation (45) has no zero row (Ref 1:444).

Two other methods for finding the gain matrix for nonsquare B matrices were used. These vary in the way a solution to equation (45) is found. An optimum, but not exact, solution is obtained using the generalized inverse of B in equation (46) for the standard inverse. In this problem, any of the methods of obtaining the generalized inverse in the next section may be used. An exact solution can be obtained using the method of singular value decomposition and its resulting factorization of B into square, invertible matrices and a matrix of singular values.

The optimum solution is limited in that it results in the construction of a least squares "best" gain matrix. This gain does not predictably move a specific system pole to the desired location. It moves all system poles in such a way as to minimize the error between the desired and actual pole location. The result is that the system may or may not be stabilized.

The exact solution is highly desireable but the application of this technique is limited since a controllability condition must be met (see Section V). As a result, the exact solution method may not control the system.

# IV. GENERALIZED INVERSE

## Moore-Penrose Pseudoinverse

A method for finding the generalized inverse of an mxn matrix B was discovered by the mathematicians Moore and Penrose earlier this century. Their simple formula found the optimum solution of the matrix equation

$$[A]x = b \tag{47}$$

for the case when A was not invertible by standard means, specifically when A was not square. In other words this method allows us to "invert" a matrix that had rank, r, equal to its smallest dimension, either m or n.

The Moore-Penrose formula for the generalized inverse is

$$B^{+} = (B^{T}B)^{-1}B^{T} \tag{48}$$

where $B^T$ is the standard transpose of B and $(B^TB)^{-1}$ is the inverse of the nxn square matrix of full rank found by performing the multiplication indicated. When B itself is square, $B^{+} = B^{-1}$.

Different formulations were developed for B having a greater number of rows, m, than columns, n, (m>n) and for the case of m<n.

The set up of the problem in this paper results in m always being greater than n, so only this case will be discussed. References 6 and 13 detail the procedures for the generalized inverse when m<n.

A factorization of B which allows us to create a generalized inverse for the case when the rank of B is less than the smaller of m or n has been developed. This variation of the Moore-Penrose generalized is termed the pseudoinverse by Strang (Ref 13) and is used for examples where $B^TB$ is singular and therefore not invertible.

$$B^+ = U'^T(U'U'^T)^{-1}(L^TL)^{-1}L'^T \ , \tag{49}$$

and a proof is found in Strang (Ref 13).

The terms L' and U' are from the factorization of B

$$B = L'U', \tag{50}$$

and the superscripts T and -1 denote the standard matrix transpose and inverse, respectively. The matrices B, L', U' are sequentially, mxn, mxr, rxn with all three having rank, r. Matrix U is formed first by Gauss reduction of B to upper triangular form. When the reduction is done, U has rank r with the bottom m-r rows being zero. We form U' by

simply eliminating the zero rows of U. Gauss elimination is accomplished by performing elementary row operations on B. This is actually done by premultiplying elementary matrices onto B (Ref 13:20-21). The matrix L is formed by multiplying the inverses of the elementary matrices in the reverse order that the row operations were performed. We reduce L to L' by eliminating the last m-r columns of L. This is possible since only the last m-r rows of U would multiply these and they are zero.

<u>Singular Value Decomposition</u>

The requirement to invert non-square real matrices has lead to the development of several techniques now in wide use. One of the lesser used, though, is singular value decomposition. An important point of this technique is that it allows us to invert a rectangular matrix of less than full rank. A matrix such as this would occur if we decided to not apply control to one mode leaving a row of zeroes in B. Likewise, as with the pseudoinverse, methods have been developed for systems in which B has a greater number of rows than columns and also for the case where $m<n$. Again, this paper always has m greater than n so the reader is referred to reference 7 when dealing with the case of $m<n$.

In general, any mxn matrix B, with $m>n$, can be factored in the form

$$[B]\ [LE][\sigma]\ [RE^T] = [LE]\begin{bmatrix}D\\0\end{bmatrix}[RE^T] \qquad (51)$$

and the generalized inverse is defined as

$$B^{+} = [RE]\begin{bmatrix} D^{+} \\ \\ 0 \end{bmatrix}[LE^{T}] \qquad (52)$$

It is constructed using the following details. The matrix LE is an mxm orthogonal matrix and its columns are the eigenvectors of $BB^T$. Similarly, the columns of RE are the eigenvectors of $B^TB$ and RE is nxn. Both RE and LE are full rank. The term $\theta$ is composed of two submatrices, the diagonal D matrix containing the m singular values and the remaining submatrix is mx(n-m) with zero elements. If the rank of the matrix is r then there are m-r singular values that are zero. The singular values are the positive square roots of the eigenvalues of $BB^T$, that is,

$$\zeta_i = +[\zeta_i(BB^T)]^{1/2} \; , \; i = 1,2,\ldots,m \qquad (53)$$

therefore,

$$D = \mathrm{diag}\ (\zeta_1, \zeta_2, \ldots \zeta_r, 0, \ldots 0) \qquad (54)$$

In constructing the generalized inverse, $B^+$, the only step of equation (52) needing clarification is $D^+$. Since D is diagonal its inverse is simply

$$D^{+} = \text{diag}\ (1/\zeta_1,\ 1/\zeta_2,\ldots,1/\zeta_r). \qquad (55)$$

The generalized inverse satisfies the following four conditions with B

$$\begin{aligned} (B^{+}B)^{T} &= B^{+}B \\ (BB^{+})^{T} &= BB^{+} \\ BB^{+}B &= B \\ B^{+}BB^{+} &= B^{+} \end{aligned} \qquad (56)$$

Specific properties of the generalized inverse are found in references 6, 7, and 13. The use of the singular value decomposition towards this problem along with numerical examples are given later.

# V. CONTROLLABILITY CONDITION

In the section on pole placement, controllability was discussed briefly. We are interested in solving (45)

$$[B]\,[G] = [\Lambda(t)]\left\{[K] - [J]\right\}[\Lambda^{-1}(t)]$$

for the gain matrix which will allow us to predictably relocate system poles. The controllability statement is the restriction on the extent to which we can predict pole location and place poles as desired.

When B has full rank the solution to (45) is straightforward and G is found at any time. The controllability is that the product [B] [G] have no zero rows.

For the case where B is nonsquare with m>n, the method of singular value decomposition can lead to an exact expression for the gain provided a consistency check (controllability condition) is met. In the previous chapter, B was factored in (51) to

$$[B] = [LE]\begin{bmatrix} D \\ \\ 0 \end{bmatrix}[RE^T].$$

This result is substituted into equation (45) giving

$$[LE]\begin{bmatrix} D \\ 0 \end{bmatrix}[RE^T]\,[G(t)] = [\Lambda(t)]\left\{[K] - [J]\right\}[\Lambda^{-1}(t)] \qquad (57)$$

Since the matrix LE is of full rank it is invertible. It is also orthonormal and its inverse is equal to its transpose. Premultiplying both sides of (57) by LE transpose results with

$$\begin{bmatrix} D \\ 0 \end{bmatrix} [RE^T]\ [G(t)] = [LE^T]\ [\Lambda(t)] \left\{ [K] - [J] \right\} [\Lambda^{-1}(t)] = [F(t)] \qquad (58)$$

The matrices F and G can be written in vector form

$$[F(t)] = [\bar{f}(t)_1,\ \bar{f}(t)_2, \ldots,\ \bar{f}(t)_m] \qquad (59)$$

and

$$[G(t)] = [\bar{g}(t)_1,\ \bar{g}(t)_2, \ldots,\ \bar{g}(t)_m] \qquad (60)$$

Equating the columns of (58)

$$\begin{bmatrix} D \\ 0 \end{bmatrix} [RE^T]\ \bar{g}(t)_1 = \bar{f}(t)_i,\ i=1,m \qquad (61)$$

We can solve for the columns of the gain matrix from

$$\bar{g}\ (t)_i = \sum_{j=1}^{m-r} \frac{1}{\sigma_j}\ (\bar{LE}_j \cdot \bar{f}(t)_i)\bar{RE}_j \qquad (62)$$

which is from Noble (Ref 6:338). This solution is found provided

$$\bar{LE}_j \cdot \bar{f}_i = 0 \quad \text{for } j=m-r,\ldots,m \quad \text{and } i=1,m. \tag{63}$$

This is also the condition for when the system is controllable.

# VI. RESULTS, CONCLUSIONS, AND RECOMMENDATIONS

The overall scheme for implementing the controller was identical for the full rank B matrix, least squares method for nonsquare B matrix, and exact singular valued decomposition of B techniques in determining the gain matrix. Once the gain matrix was obtained by one of these methods, equations (40) and (35) were used to generate equation(34), which was then solved numerically. The Poincaré exponents were then obtained and system stability and pole location checked.

The first set of cases examined in the stabilization of a satellite in periodic orbit about $L_3$ was for B having full rank. Here, B was a six-by-six identity matrix allowing control to be applied to all state variables equally. The gain matrix as a function of time was calculated from (46) for a selection of pole locations. The values of desired pole location as well as the resultant Poincaré exponents are tabulated in Table II. Values for desired pole location were picked as representative of what could typically be chosen. In the three cases done, pole location was exactly as desired. As predicted, with all modes controllable, placement of poles was straightforward.

This result is important for a periodic system where all the states are controllable. The technique results in stable control in a predictable manner. In the physical terms for the restricted four-body

problem, however, this case cannot be implemented as it requires instantaneous control over satellite position (teleportation) and we can only control momentum.

TABLE II

Normalized Poincaré Exponents, Desired Location and Resultant Location for Square, Full Rank B Matrix

| Case | Desired Location | | Resultant Location | |
|---|---|---|---|---|
| Move unstable | 0.0 | + 0.077738i | 0.0 | + 0.077738i |
| real pole | 0.0 | - 0.077738i | 0.0 | - 0.077738i |
| | -0.1 | + 0.0i | -0.1 | + 0.0i |
| | -0.17893 | + 0.0i | -0.17893 | + 0.0i |
| | 0.0 | + 0.084134i | 0.0 | + 0.084134i |
| | 0.0 | - 0.084134i | 0.0 | - 0.084134i |
| Move unstable | -0.1 | + 0.077738i | -0.1 | + 0.077738i |
| real pole, | -0.1 | - 0.077738i | -0.1 | - 0.077738i |
| damp first | -0.1 | + 0.0i | -0.1 | + 0.0i |
| oscillatory | -0.17893 | + 0.0i | -0.17893 | + 0.0i |
| mode | 0.0 | + 0.084134i | 0.0 | + 0.084134i |
| | 0.0 | - 0.084134i | 0.0 | - 0.084134i |

TABLE II (Cont'd.)

| Case | Desired Location | Resultant Location |
|---|---|---|
| Move unstable | -0.1 + 0.077738i | -0.1 + 0.077738i |
| real pole, | -0.1 - 0.077738i | -0.1 - 0.077738i |
| damp both | -0.1 + 0.0i | -0.1 + 0.0i |
| oscillatory | -0.17893 + 0.0i | -0.17893 + 0.0i |
| modes | -0.1 + 0.084134i | -0.1 + 0.084134i |
| | -0.1 - 0.084134i | -0.1 - 0.084134i |

The B matrix which allows control of only momentum in the restricted four-body problem is given by

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Its generalized inverse was found both by the Moore-Penrose and singular value decomposition methods to be

$$[B^+] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Use of this result directly in (45) for $B^{-1}$ gave the time varying gain matrix which, when implemented into the system, minimized the error between desired and resultant pole location. The case considered was the repositioning of the unstable real root to the same location as was used for the full rank B system. The normalized Poincaré exponents desired and those resulting from this method are listed in Table III.

TABLE III

Normalized Poincaré Exponents, Desired Location and Resultant Location for Nonsquare B Matrix Using Generalized Inverse Method of Gain Calculation

| Desired Location | Resultant Location |
|---|---|
| 0.0 + 0.077738i | -0.0845 + 0.1204i |
| 0.0 - 0.077738i | -0.0845 - 0.1204i |
| -0.1 + 0.0i | 0.0492 + 0.1979i |
| -0.17893 + 0.0i | 0.0492 - 0.1979i |
| 0.0 + 0.084134i | 0.0 + 0.084134i |
| 0.0 - 0.084134i | 0.0 - 0.084134i |

The desired relocation was not obtained. Both the real mode and first oscillatory mode were assigned new values such that the system is still unstable. The form of the gain matrix and coupling between planar modes resulted in unpredictable resultant pole location. The second oscillatory mode was unchanged since the gain matrix had no corresponding elements to effect it. Also, this out of plane mode is uncoupled from the planar modes (Ref 9) and hence suffered no cross-feed from the others.

The final case examined was that of calculating the gain matrix as a function of time based upon the technique of Chapter V. The singular value decomposition of the nonsquare B matrix was obtained and used to get (58). Only relocation of the unstable real root was attempted. For each type increment in the period, (63) was evaluated to check controllability using an m of six and r of 3. The values from (63) filled a three-by-six matrix which was inspected for nonzero elements. For each increment of time, one of the three columns in the matrix was nonzero. The controllability condition was never met so no exact solution was obtainable. This was verified by calculating the Poincaré exponents of the controlled system using the gain from (62). Table IV lists the values obtained.

TABLE IV

Poincaré Exponents for System with

Unmet Controllability Condition

| Desired Location | | | Resultant Location | | |
|---|---|---|---|---|---|
| 0.0 | + | 0.077738i | -0.000393 | + | 0.077726i |
| 0.0 | - | 0.077738i | -0.000393 | - | 0.077726i |
| -0.1 | + | 0.0i | 0.19522 | + | 0.0i |
| -0.17893 | + | 0.0i | -0.17876 | + | 0.0i |
| 0.0 | + | 0.084134i | 0.00011 | + | 0.08231i |
| 0.0 | - | 0.084134i | 0.00011 | - | 0.08231i |

From this table it is seen that each mode was effected. The fact that the unstable real root maintained its positive value (it actually became more positive) verifies the uncontrollable nature of the system using pole placement. No case was developed where it was thought the controllability condition would meet.

The use of pole placement in time periodic systems is highly effective when all of the states are controllable. Predictable placement of poles was achieved for all cases in a completely controllable system. For systems with uncontrollable states, such as positions in the restricted four-body problem, predictable placement of system poles was not achieved.

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# VITA

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER AFIT/GAE/AA/82D-33 | 2. GOVT ACCESSION NO. AD-A124 037 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) POLE PLACEMENT FOR TIME PERIODIC SYSTEMS | | 5. TYPE OF REPORT & PERIOD COVERED MS THESIS |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) CARL S. WOLF CAPT, USAF | | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology (AFIT/EN) Wright-Patterson AFB, OH 45433 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS | | 12. REPORT DATE December 1982 |
| | | 13. NUMBER OF PAGES 48 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report) Unclassified |
| | | 15a. DECLASSIFICATION DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release: IAW AFR 190-17.

LYNN E. WOLAVER
Dean for Research and Professional Development
Air Force Institute of Technology (ATC)
Wright-Patterson AFB OH 45433

19 JAN 1983

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Periodic Orbit
Lagrange Points
Libration Points
Pole Placement
Poincare Exponents
Generalized Inverse
Singular Value Decomposition

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The technique of pole placement was applied to a time varying periodic system. When the control matrix was of full rank, unstable real roots were predictably moved to stable locations. Specified damping was successfully applied to oscillatory roots. In the case of the control matrix having less than full rank a least squares gain was applied. This minimized the error between all desired pole locations and controlled pole locations, and, as a consequence, prediction of final pole location was not

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possible. An exact method for predictably placing poles as desired was restricted by a controllability condition.

The method was applied to the restricted four-body problem and control of a satellite in a periodic orbit about the Earth-Moon Lagrangian point, L3. A stabilized orbit was obtained for a model which allowed instantaneous control of both position and velocity. Stabilization was not successful in the case of control of velocity only. The least sqares solution effected several modes simutaneously. The exact solution for gain was unobtainable as the controllability condition was not met.

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